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The effect of the boundary condition at a fin tip on the performance of the fin with and without internal heat generation

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Abstract—An analytical investigation is carried out on the effect on the rate of heat flow from a one-dimensional straight (or cylindrical) fin with and without internal heat generation of the assumption that no heat transfer takes place at the fin tip. This assumption yields trivial solutions for some values of the parameters considered if internal heat sources exist in the fin. The error in the determination of the foregoing rate of heat flow appears to be large for some conditions important for practical applications. For the quantitative analysis of the fin, the following parameter ranges are used: Biot number, 0.1 – any value lower than 0.1; generation number, 0–0.5; and the ratio of the length of the fin to its half-thickness (or its half-radius), 1–100. The heat transfer coefficient is assumed to be a power function of the temperature difference between the fin and its surroundings and that the power in this function is equal to −1, 0, 1 and 2.

INTRODUCTION

THE OBJECT of this study has been to analytically investigate the effect on the performance of a straight fin of rectangular profile (or a cylindrical fin) with and without internal heat generation of the boundary condition at the fin tip. To this end, a one-dimensional analysis has been carried out. This is justified if the Biot number is less than 0.1. In this case the error made in the determination of the rate of heat transfer from the fin to the fluid surrounding it is less than 1% [1, 2].

Since the one-dimensional differential equation of the temperature distribution in the fin is of second order, two boundary conditions are required to solve this differential equation. The boundary condition at the fin base is that the temperature there is constant. For the fin tip, one of the two boundary conditions can be selected, the condition that no heat transfer takes place at the fin tip or the condition that heat transfer takes place at the fin tip. Herein, the first mentioned boundary condition at the fin tip is referred to as the hypothetical boundary condition and the latter as the real boundary condition.

For the steady-state one-dimensional analytic analysis of a straight (or a cylindrical) fin with no internal heat sources, a uniform heat transfer coefficient and the hypothetical boundary condition at the fin tip were considered in practically all the classical references [3–7] and in handbooks [8, 9]. In a few studies, the heat transfer coefficient was assumed to be dependent either on the space coordinate [10–13] or the temperature of the fin itself [14–17]. In refs. [18, 19], the real boundary condition at the fin tip was used however and the heat transfer coefficient was assumed to depend on the temperature of the fin itself.

A uniform heat transfer coefficient and the hypothetical boundary condition at the fin tip were also used in the analytic one-dimensional analysis of a straight (or a cylindrical) fin with internal heat generation [20–23]. In refs. [24, 25], the heat transfer coefficient was taken as a function of the temperature of the fin.

For most industrial applications, the heat transfer coefficient is given by

$$h = a\theta^n \tag{1}$$

where a and n are constants. The dimensionless constant, n, in equation (1), may vary in general between approximately −6.6 and 5 and in most practical applications between approximately −3 and 3. If the heat transfer coefficient is given by equation (1) and the hypothetical boundary condition at the fin tip is taken into account, then the one-dimensional differential equation of the temperature distribution in a straight (or a cylindrical) fin with no internal heat sources can be analytically solved only for a few values of n in equation (1) using ordinary (i.e. algebraic, logarithmic and circular) functions [14]. These values of n are equal to 0, −1.0, −1.5, −1.6, −1.8, −1.9, −2.1, −2.2, −2.4, −2.5, −3.0 and −4.0 if n is either an integer or a one-digit number. For the remaining values of n, the temperature distribution in the fin is expressed in a transcendental function, i.e. in Legendre's normal elliptic integral of the first kind for $n = 1$ and 2 [14] and in the hypergeometric function for $-\infty < n < \infty$ [26].

The contents of this paper are outlined below. Considering that the real boundary condition at the fin tip and taking $n = -1$, 0, 1 and 2, respectively, then the

NOMENCLATURE

A	cross-sectional area of a fin [m^2]
a	given constant [$W m^{-2} K^{-(n+1)}$]
$a_1, \ldots, a_9$	constants defined in the text
Bi	modified Biot number at the base of a fin
b	modified fin parameter
e	error
$F(\mu/\alpha)$	Legendre's normal elliptic integral of the first kind
f	fin effectiveness
$g, \ldots, g_2$	constants defined in the text
h	heat transfer coefficient [$W m^{-2} K^{-1}$]
i	imaginary number, $\sqrt{-1}$
$J_1, \ldots, J_6$	constants or functions defined in the Appendix
K	thermal conductivity of the fin material [$W m^{-1} K^{-1}$]
L	fin length [m]
M, m	constants defined in the text
N	generation number
n	given constant
P	circumference of a cylindrical fin [m]
Q	internal heat generation [$W m^{-3}$]
q	rate of heat transfer from a fin [W]
S	dimensionless rate of heat transfer from a fin
T	dimensionless temperature
U	half-fin thickness [m]
W	constant defined in the text [m]
X	dimensionless space coordinate
x	space coordinate [m]
Y, Z	constants defined in the text.

Greek symbols

α	modular angle [rad]
β	root of a polynomial equation
γ	constant defined in the text
θ	difference between the temperature of a fin and that of the fluid surrounding it at point x [K]
μ	amplitude [rad].

Subscripts

$1, \ldots, 4$	order of a root of a polynomial
b	fin base (i.e. $x = 0$)
e	fin tip (i.e. $x = L$)
h	hypothetical boundary condition at the fin tip
m	maximum value
r	reduced value
t	real boundary condition at the fin tip.

analytic solutions of the one-dimensional differential equation of the temperature distribution in a straight (or a cylindrical) fin with uniform internal heat generation are presented. These solutions are in a general form and apply also to the case where no internal heat sources exist in the fin and also to the case where no heat transfer takes place at the fin tip. The dimensionless rate of heat transfer from the fin to the fluid surrounding it is taken as a criterion for characterizing the performance of the fin. This criterion can be determined if the temperature distribution in the fin is known. For given values of n, the modified Biot number Bi, the generation number N and the ratio of the length of the fin to its half-thickness (or half-radius) L/W and using the real and hypothetical boundary condition, respectively, the foregoing criterion can be calculated and the values compared to one another. The ranges of the dimensionless variables considered are: $n = -1$, 0, 1 and 2; $Bi = 0.1$ – any value lower than 0.1; $N = 0$–0.5 and $L/W = 1$–100. For some conditions important for practical applications however, the error in the prediction of the rate of heat transfer from the fin to its surroundings appears to be of a significant magnitude if it is assumed that no heat transfer takes place at the fin tip. Furthermore if $n = 1, 2$ and $N > 0$, then the solutions obtained with this assumption are trivial beyond a certain value of L/W for a given value of Bi.

To the author's knowledge and in the cases where $n = 1, 2$ and $N > 0$, then the analytic solutions of the foregoing non-linear differential equation are not available in the literature if the real boundary condition at the fin tip is being considered. These solutions may also be of importance in nuclear and chemical engineering since this differential equation applies to diffusion and chemical reaction provided that the appropriate changes in notation, the assumptions and the remaining boundary condition are taken into account.

Some of the solutions presented herein include Legendre's normal elliptic integral of the first kind. Simple formulae are given to calculate this elliptic integral in the Appendix.

DIFFERENTIAL EQUATION OF TEMPERATURE DISTRIBUTION

A straight fin (of rectangular profile) or a cylindrical (i.e. pin) fin is now considered. For the analysis of such a fin, the following assumptions are made: one-dimensional steady-state heat conduction through the fin, a constant thermal conductivity of the fin material and a constant cross-sectional area for the fin. Internal heat generation in the fin is either uniform or zero. The temperature of the fluid surrounding the fin is constant. The origin of the space coordinate x is at the fin base and the positive value of x is toward the fin tip. The straight fin is infinitely long in the longitudinal direction. A unit length of this fin in this direction is being considered here. The heat transfer coefficient is given by equation (1).

For the assumptions being made, the non-dimensional differential equation of the temperature distribution in the fin then becomes

$$\frac{d^2T}{dX^2} - bT_b^{-n}T^{(n+1)} = -bNT_b. \quad (2)$$

The boundary conditions are expressed by

$$T = T_b = \theta_b/\theta_e \quad \text{for } X = 0 \quad (3)$$

and

$$\frac{dT}{dX} = -Bi\frac{L}{W}T_b^{-n} \quad \text{for } X = 1. \quad (4)$$

The non-dimensional parameters used in equations (2)–(4) are defined as follows:

$$T = \theta/\theta_e \quad (5)$$

$$X = x/L \quad (6)$$

$$b = Bi\left(\frac{L}{W}\right)^2 \quad (7)$$

$$Bi = \frac{h_b W}{K} = \frac{a\theta_b^n W}{K} \quad (8)$$

$$N = \frac{QW}{h_b\theta_b} = \frac{QW}{a\theta_b^{(n+1)}} \quad (9)$$

in which

$$W = U \quad \text{for the straight fin} \quad (10)$$

and

$$W = A/P \quad \text{for the cylindrical fin.} \quad (11)$$

In accordance with the first boundary condition expressed in equation (3), the temperature difference at the fin base is equal to θ_b. The second boundary condition in equation (4) implies that heat transfer takes place at the fin tip, i.e. $-K(d\theta/dx) = h_e\theta_e$ for $x = L$. For the hypothetical boundary condition at the fin tip equation (4) reduces to

$$\frac{dT}{dX} = 0 \quad \text{for } X = 1. \quad (12)$$

Equation (12) expresses the fact that no heat transfer takes place at the fin tip or that the fin tip is perfectly insulated. Strictly speaking, this condition can be only realized if the fin is infinitely long.

The definition of the Biot number given in the relative literature is identical to that of the modified Biot number used herein (see equation (8)) for the straight fin but is different for the cylindrical fin. For the latter, the modified Biot number is half of the Biot number defined in the literature.

In order to solve equation (2), a and n in equation (1), the fin thickness $2U$ (or the fin diameter), the amount of internal heat generation Q, the temperature difference between the fin and its surroundings at the fin base θ_b and the thermal conductivity of the fin material K should be known. These values have been reduced to three dimensionless variables, i.e. the modified Biot number Bi, the ratio of the length of the fin to its half-thickness (or half-radius) L/W and the generation number N, as can be deduced from equations (7) to (11). In equation (2) T_b is not an unknown since it is determined with the solution of this equation considering that $T_b = \theta_b/\theta_e$ and that θ_e is the temperature difference between the fin and its surroundings at the fin tip.

The ratio of the interior to exterior resistances for the fin is characterized by Bi and the fin shape by L/W. The ratio of the total heat generated in the fin to the heat that would be dissipated from the fin if all of the fin was at the base temperature and that no heat transfer took place at the fin tip is N [20]. If no heat transfer takes place at the fin tip then the definition of N implies that the maximum value of N is equal to 1 and that no heat is conducted into the fin at its base for $N = 1$ since $dT/dX = 0$ along the fin. If now the heat transfer takes place at the fin tip then N has no physical meaning but it is still a convenient dimensionless variable for the analysis of the fin since, per definition, dT/dX cannot be zero along the fin (see equation (4)). The maximum value of this generation number, N_m, for the condition that heat transfer takes place at the fin tip, can be determined with a trial and error method, as will become clear to the reader later.

SOLUTION OF DIFFERENTIAL EQUATION

In order to solve equation (2) using the real boundary condition at the fin tip and the boundary condition expressed in equation (3) for $n = -1$, 0, 1 and 2, the procedures given in refs. [17, 25] have been used and in which the solutions of this equation with the hypothetical boundary condition at the fin tip were given. Therefore, it seems sufficient to present only the solutions of equation (2) but not the details in these solutions.

In the cases where $n = -1$ and 0

The dimensionless temperature in the fin, T, and the dimensionless temperature at the base of the fin, T_b, for $n = -1$ are given by

$$T = T_b\left[\frac{b}{2}\left\{(1-N)X^2 - 2\left(1 - N + \frac{W}{L}\right)X\right\} + 1\right] \quad (13)$$

$$T_b = \left\{1 - \frac{b}{2}\left(1 - N + 2\frac{W}{L}\right)\right\}^{-1} \quad (14)$$

and those for $n = 0$ by

$$T = NT_b - \sqrt{Bi}\,e^{-\sqrt{b}(1-X)} + (T_b - NT_b + \sqrt{Bi}\,e^{-\sqrt{b}}) \times \frac{\cosh\{\sqrt{b}(1-X)\}}{\cosh\sqrt{b}} \quad (15)$$

$$T_b = \frac{1+\sqrt{Bi}\{1-(e^{\sqrt{b}}\cosh\sqrt{b})^{-1}\}}{N+(1-N)/\cosh\sqrt{b}}. \quad (16)$$

If no internal heat sources exist in the fin, N is taken as being equal to zero in equations (13)–(16) (i.e. $N = Q = 0$).

If the hypothetical boundary condition at the fin tip is considered, then W/L equations (13) and (14) and the terms including Bi in equations (15) and (16) disappear since T_b is a function of b and N, and T a function of b, N and X, as equations (2), (3) and (12) imply. The foregoing can also be explained as follows: for a given value of b, numerous combinations of the values of Bi and L/W are possible (see equation (7)), but for given values of b and L/W, then the value of Bi is unique. If L is increased (keeping W, N and b constant) both W/L and Bi decrease. For sufficiently large values of L, the rate of heat transfer from the tip of the fin should be negligible when compared with that from the fin, consequently the boundary condition at the fin tip should not affect T and T_b and the values of Bi and W/L should be very small. Accordingly, the terms including W/L in equations (13) and (14) and those including Bi in equations (15) and (16) become negligibly small if the fin is very long, a condition which practically implies the hypothetical boundary condition at the fin tip.

In the cases where $n = 1$ and 2

In order to calculate the dimensionless temperature distribution in the fin for these cases, equation (2) should be integrated twice. The first integration of this equation including the determination of the integration constant (using the real boundary condition at the fin tip) is presented below for all values of n excluding -2

$$\frac{dT}{\left\{\gamma\left[T^{(n+2)} - (n+2)NT_b^{(n+1)}T + (n+2)NT_b^{(n+1)} - 1 + \frac{(n+2)Bi}{2T_b^n}\right]\right\}^{0.5}} = -dX \quad (17)$$

where

$$\gamma = \frac{2bT_b^{-n}}{n+2}. \quad (18)$$

The integration of equation (17) including the calculation of the integration constant (using the boundary condition expressed in equation (3)) yields

$$mF(\mu/\alpha) = -X + mF(\mu_b/\alpha). \quad (19)$$

In equation (19) $F(\mu/\alpha)$ is Legendre's normal elliptic integral of the first kind. Its value can be determined if the amplitude μ and the modular angle α in it are known. In ref. [27] $F(\mu/\alpha)$ is tabulated and it can be also predicted with the simple formulae given in the Appendix. For known values of T_b, Bi, L/W and N, μ is only a function of T and α is a constant. In equation (19) m is also constant depending on the foregoing four parameters. With equation (19) T_b is solved taking $T = T_e = 1$ for $X = 1$. In this case this equation reduces to

$$mF(\mu_b/\alpha) - mF(\mu_e/\alpha) = 1. \quad (20)$$

The left-hand side of equation (20) includes only T_b, Bi, L/W and N. From this equation T_b is solved as will be explained later. In equation (20) μ_b is the value of μ for $T = T_b$ and μ_e the value of μ for $T = T_e = 1$.

In order to calculate μ, α and m in equation (19), the roots of the polynomial equation in equation (17) are required. Relevant to these roots and n, only the following conditions should be considered to determine the temperature distribution in the fin [25].

(1) For $n = 1$, the polynomial (i.e. cubic) equation has three real roots β_1, β_2 and β_3 and $1 \geqslant \beta_1 > \beta_2 > \beta_3$.

(2) For $n = 1$, the polynomial equation has one real root β_1 and two complex roots β_2 and β_3, and β_1 is a positive real number equal to 1 or smaller than 1.

(3) For $n = 2$, the polynomial (i.e. fourth order) equation has two real roots β_1 and β_2 and two complex roots β_3 and β_4, and $1 \geqslant \beta_1 > \beta_2 \geqslant -1$.

For $n = 1$ and in the case where the polynomial equation has three real roots; μ, m, α, μ_b and μ_e are given by

$$\mu = \arcsin\left\{\left(\frac{\beta_1 - T}{\beta_2 - T}\right)^{0.5}\right\} \quad \text{for } 0 \leqslant \mu \leqslant \pi/2 \quad (21a)$$

$$m = 2\{\gamma(\beta_1 - \beta_3)\}^{-0.5} \quad (22a)$$

$$\alpha = \arcsin\left\{\left(\frac{\beta_2 - \beta_3}{\beta_1 - \beta_3}\right)^{0.5}\right\} \quad (23a)$$

$$\mu_b = \arcsin\left\{\left(\frac{\beta_1 - T_b}{\beta_2 - T_b}\right)^{0.5}\right\} \quad (24a)$$

$$\mu_e = \arcsin\left\{\left(\frac{\beta_1 - 1}{\beta_2 - 1}\right)^{0.5}\right\}. \quad (25a)$$

For $n = 1$ and in the case where the polynomial equation has one real root β_1 and two complex roots β_2

and β_3; the latter are given by

$$\beta_{2,3} = r \pm Yi \tag{26}$$

and μ, m, α, μ_b and μ_e by

$$\mu = \arccos\left(\frac{T-r-M\cot Z}{T-r+M\tan Z}\right) \quad \text{for } 0 \leqslant \mu \leqslant \pi \tag{21b}$$

$$m = -M(\tan Z+\cot Z)\left\{\frac{4\gamma M^3}{\sin^3(2Z)}\right\}^{-0.5} \tag{22b}$$

$$\alpha = Z \tag{23b}$$

$$\mu_b = \arccos\left(\frac{T_b-r-M\cot Z}{T_b-r+M\tan Z}\right) \tag{24b}$$

$$\mu_e = \arccos\left(\frac{1-r-M\cot Z}{1-r+M\tan Z}\right) \tag{25b}$$

in which

$$M = |Y| \tag{27}$$

$$\tan(2Z) = \frac{M}{\beta_1 - r} \quad \text{for } 0 < 2Z < \pi. \tag{28}$$

For $n = 2$ and if the polynomial equation has two real roots β_1 and β_2 and two complex roots β_3 and β_4; these complex roots are given by

$$\beta_{3,4} = a_1 \pm gi \tag{29}$$

and μ, m, α, μ_b and μ_e by

$$\mu = \arccos\left(\frac{a_8-a_6T}{a_5T-a_7}\right) \quad \text{for } 0 \leqslant \mu \leqslant \pi \tag{21c}$$

$$m = \frac{2a_4a_9}{\{-2\gamma a_2a_4(\beta_1-\beta_2)\}^{0.5}} \tag{22c}$$

$$\alpha = \arcsin a_9 \tag{23c}$$

$$\mu_b = \arccos\left(\frac{a_8-a_6T_b}{a_5T_b-a_7}\right) \tag{24c}$$

$$\mu_e = \arccos\left(\frac{a_8-a_6}{a_5-a_7}\right) \tag{25c}$$

in which

$$a_2 = |g| \tag{30}$$

$$a_3 = \frac{a_2^2+(\beta_1-a_1)(\beta_2-a_1)}{a_2(\beta_1-\beta_2)} \tag{31}$$

$$a_4 = a_3-(a_3^2+1)^{0.5} \tag{32}$$

$$a_5 = \beta_1-a_1-a_2/a_4 \tag{33}$$

$$a_6 = \beta_1-a_1+a_2a_4 \tag{34}$$

$$a_7 = \tfrac{1}{2}(\beta_1+\beta_2)a_5-\tfrac{1}{2}(\beta_1-\beta_2)a_6 \tag{35}$$

$$a_8 = \tfrac{1}{2}(\beta_1+\beta_2)a_6-\tfrac{1}{2}(\beta_1-\beta_2)a_5 \tag{36}$$

$$a_9 = (1+a_4^2)^{-0.5}. \tag{37}$$

If no heat sources exist in the fin (i.e. $Q = N = 0$) then the roots of the foregoing polynomial equation can be given in simple expressions, and equations (21)–(25) can be significantly simplified. Since the design engineer deals mostly with the applications in which $N = 0$, it is considered justifiable to present these roots and simplified equations for $N = 0$.

In the case where $n = 1$ and $N = 0$, the polynomial equation has one real root β_1 and two complex roots β_2 and β_3 and these roots are given by

$$\beta_1 = \left(1-\frac{3}{2}\frac{Bi}{T_b}\right)^{1/3} \tag{38}$$

$$\beta_{2,3} = \frac{\beta_1}{2}(-1 \pm i\sqrt{3}) \tag{39}$$

and μ, m, α, μ_b and μ_e are given by

$$\mu = \arccos\left(\frac{T+\beta_1 g_1}{T+\beta_1 g_2}\right) \quad \text{for } 0 \leqslant \mu \leqslant \pi \tag{21b$'$}$$

$$m = -\frac{\sqrt{3}}{2}\beta_1(\tan Z+\cot Z)\left(\frac{3\sqrt{3}\gamma\beta_1^3}{2\sin^3 2Z}\right)^{-0.5} \tag{22b$'$}$$

$$\alpha = Z = \pi/12 \tag{23b$'$}$$

$$\mu_b = \arccos\left(\frac{T_b+\beta_1 g_1}{T_b+\beta_1 g_2}\right) \tag{24b$'$}$$

$$\mu_e = \arccos\left(\frac{1+\beta_1 g_1}{1+\beta_1 g_2}\right) \tag{25b$'$}$$

in which

$$g_1 = \tfrac{1}{2}(1-\sqrt{3}\cot Z) \tag{40}$$

$$g_2 = \tfrac{1}{2}(1+\sqrt{3}\tan Z). \tag{41}$$

As noted earlier herein, β_1 given by equation (38) should always be a positive real number equal to 1 or smaller than 1; accordingly $3Bi/(2T_b)$ should always be smaller than 1. This can be explained as follows: if $3Bi/(2T_b)$ is greater than 1, then the real root of the polynomial equation is given by

$$\beta_1 = -\left(\frac{3}{2}\frac{Bi}{T_b}-1\right)^{1/3} \tag{42}$$

and the complex roots can be given by equation (39) if β_1 expressed in equation (42) is considered. It now follows from equations (26) to (28) and equations (39) and (42) that $\tan 2Z$ is always negative, contrary to its definition. Therefore, if $3Bi/(2T_b) > 1$, the fin fails to operate and accordingly the temperature distribution in it cannot be determined. The foregoing also implies that the polynomial equation has always one real and two complex roots if $n = 1$ and $N = 0$.

For $n = 2$ and $N = 0$, the polynomial equation has two real roots β_1 and β_2 and two complex roots β_3 and β_4, and these roots are expressed in

$$\beta_{1,2} = \pm\left(1 - \frac{2Bi}{T_b^2}\right)^{1/4} \quad (43)$$

$$\beta_{3,4} = \pm\beta_1 \mathrm{i} \quad (44)$$

and μ, m, α, μ_b and μ_e in

$$\mu = \arccos\left(-\frac{\beta_1}{T}\right) \quad \text{for } 0 \leqslant \mu \leqslant \pi \quad (21c')$$

$$m = -\frac{1}{\beta_1\sqrt{(2\gamma)}} \quad (22c')$$

$$\alpha = \pi/4 \quad (23c')$$

$$\mu_b = \arccos(-\beta_1/T_b) \quad (24c')$$

$$\mu_e = \arccos(-\beta_1). \quad (25c')$$

The expression in parentheses in equation (43) cannot be negative; thus $2Bi/T_b^2$ should always be smaller than 1. If this expression is negative, then the polynomial equation has four complex roots and accordingly the fin fails to function. This will be further explained when discussing the temperature profiles in the fin for $n = 1$ and 2.

Relative to equations (21)–(25), the a-versions of the equations are valid if the polynomial equation has three real roots and $n = 1$; the b-versions of the equations are valid if the polynomial equation has one real and two complex roots and $n = 1$. The c-versions apply to $n = 2$. The b′- and c′-versions apply to $n = 1$ and 2, respectively, if $N = 0$.

If the hypothetical boundary condition at the fin tip is considered, the term including Bi in the polynomial equation disappears. Provided that this modified form of the polynomial equation is used, equations (19)–(37) are valid. If $N = 0$, the term including Bi disappears in equations (38) and (43), the remaining equation being valid (excluding equation (42)). For the hypothetical boundary condition, the roots of the polynomial equation are expressed in simple formulae for $N > 0$ in ref. [25].

In order to determine the temperature distribution in the fin for given values of n (i.e. $n = 1$ or 2), Bi, L/W and N (i.e. $0 \leqslant N \leqslant N_m$) first T_b is determined with equation (20). To this end, a value for T_b is assumed (i.e. $T_b > 1$). The roots of the polynomial equation in equation (17) are calculated using this T_b and the other foregoing parameters. With these roots and parameters, m, α, μ_b and μ_e are predicted with equations (22)–(25), respectively. The values of $F(\mu_b/\alpha)$ and $F(\mu_e/\alpha)$ are then determined with the tabulated values of $F(\mu/\alpha)$ or with the formulae presented in the Appendix. The value of T_b is iterated until equation (20) is satisfied.

Having found T_b, m, $F(\mu_b/\alpha)$ and α, the evaluation of T for a given value of X is carried out with equation (19). First $F(\mu/\alpha)$ is predicted with this equation since X, m and $F(\mu_b/\alpha)$ in it are known. Thereafter the value of μ, which satisfies this $F(\mu/\alpha)$, is obtained with the tabulated values or with the formulae in the Appendix. Then T is determined from equation (21).

For the evaluation of X for a given value of T (i.e. $1 < T \leqslant T_b$), μ is first solved from equation (21), and thereafter $F(\mu/\alpha)$ is solved from the tabulated values or from the formulae in the Appendix, and X is finally solved from equation (19). The foregoing method seems simpler than the previous one.

ILLUSTRATION OF DIMENSIONLESS TEMPERATURE AT FIN BASE

It follows from the foregoing that it is sufficient to know n, Bi, L/W and N for the determination of T_b and N, Bi, L/W, N and X for that of T. Taking Bi and N as parameters, T_b was plotted against L/W in Fig. 1 for $n = -1$ and 0, and in Figs. 2 and 3 for $n = 1$ and 2, respectively. The ratio L/W was varied between 1 and 100; Bi was taken as being equal to values of 0.001, 0.01 and 0.1; N was equal to 0 and 0.5 for $n = -1$ and 0, and to 0, 0.25 and 0.5 for $n = 1$ and 2.

For $n = -1$, the heat flux on the surface of the fin is uniform. For this particular case, the asymptotic behaviour of T_b is obvious in Fig. 1. The value of θ_e is solved from equation (14)

$$\theta_e = \left[1 - \frac{Bi}{2}\frac{L^2}{W^2}\left(1 - N + 2\frac{W}{L}\right)\right]. \quad (45)$$

Since θ_e cannot be either negative or zero, the values of Bi and L/W are restricted in accordance with equation (45) for a given value of N (i.e. $0 \leqslant N \leqslant N_m$).

For $n = 0$, the heat transfer coefficient is uniform. For $n = 0$ and $0 < N \leqslant N_m$, the asymptotic behaviour of T_b is illustrated in Fig. 1. According to equation (16) and noting the definition of b given by equation (7), the asymptotic value of T_b (i.e. the value of T_b for large values of L/W) is expressed by

$$T_b = \frac{1 + \sqrt{Bi}}{N}. \quad (46)$$

Contrary to the case analysed for $n = -1$, no restriction applies to the magnitudes of Bi and L/W for $n = 0$ and $0 \leqslant N \leqslant N_m$.

Figures 2 and 3 show that for $N = 0$, $n = 1$ or 2 and a given value of Bi, and if $1 \leqslant L/W < \infty$, T_b increases if L/W is increased. But for given values of Bi and N (i.e. $0 < N \leqslant N_m$) and for $L/W \geqslant 1$, the fin fails to function beyond a certain value of L/W. For the values of L/W smaller than this certain value, the fin operates; thus the value of L/W is restricted. This is best explained with two examples given below. For example first consider the case where $Bi = 0.1$, $N = 0.5$, $L/W \geqslant 1$ and $n = 1$ or 2. For these values of the parameters, the fin fails to function. Consequently the temperature distribution in the fin cannot be determined with equations (19)–(25). This is due to the fact that the roots of the polynomial equation in equation (17) do not satisfy the conditions required for these roots in the case where $Bi = 0.1$, $N = 0.5$, $n = 1$ or 2 and $L/W \geqslant 1$. The curve for $N = 0.5$ and $Bi = 0.1$ is

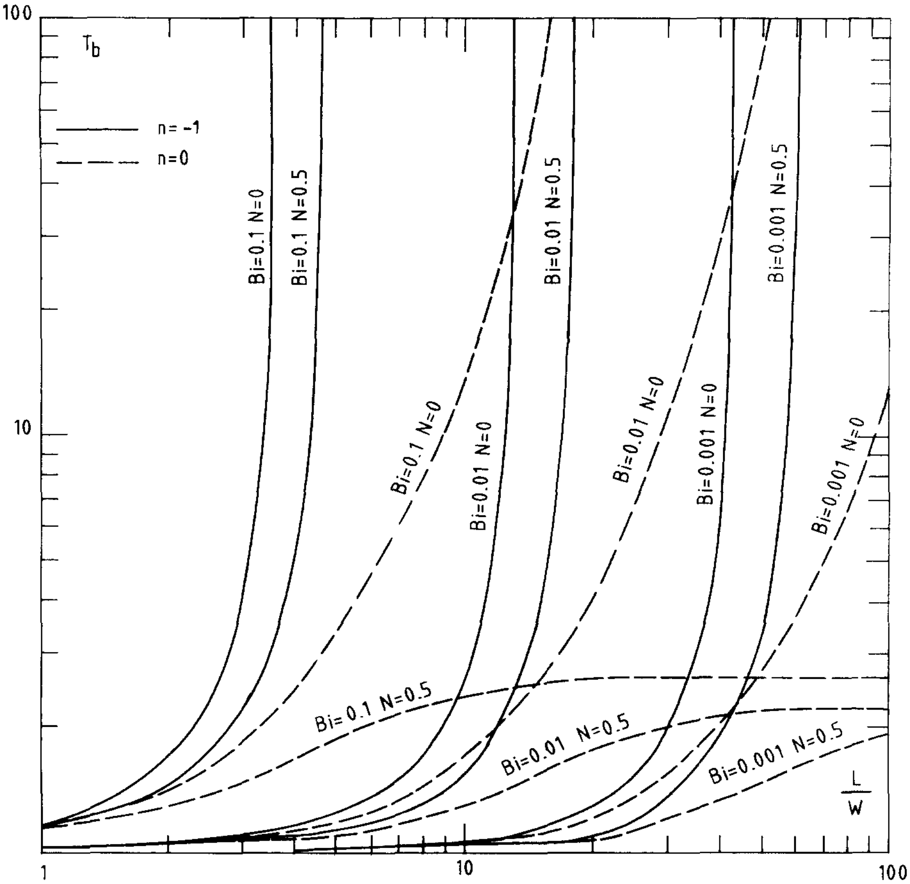

FIG. 1. Dimensionless temperature at the fin base for $n = -1$ and 0.

missing in Figs. 2 and 3, respectively, since the fin operates for $L/W \leqslant 0.768$ if $n = 1$ and it is invalid for $0 < L/W \leqslant 1$ for $n = 2$.

Secondly, consider for example the case where $Bi = 0.01$, $N = 0.25$ and $n = 1$ or 2. For these values of the parameters, the maximum value of L/W is equal to 27.58 for $n = 1$ and 22.98 for $n = 2$, as can be seen from Figs. 2 and 3. These values of L/W are rounded values. The coordinates of the end point of a discontinuous curve in these figures give the maximum values of L/W and T_b. If L/W exceeds 27.58 for $n = 1$ and 22.98 for $n = 2$, then the fin fails to function. This is explained as follows: for $n = 1$, $Bi = 0.01$, $N = 0.25$ and $1 \leqslant L/W < 27.58$, the foregoing polynomial equation has three real roots β_1, β_2 and β_3, and $1 > \beta_1 > \beta_2 > \beta_3$. For $n = 2$, $Bi = 0.01$, $N = 0.25$ and $1 \leqslant L/W < 22.98$, the polynomial equation has two real roots β_1 and β_2 and two complex roots, and $1 > \beta_1 > \beta_2 > -1$. For these quoted values of the parameters, if L/W is increased T_b increases, as shown in Figs. 2 and 3, and β_2 approaches β_1. For $n = 1$ and $L/W = 27.58$, β_2 is equal to β_1 and μ, α, μ_b and μ_e are equal to $\pi/2$ (see equations (21a) and (23a)–(25a)). If the modular angle α and the amplitude μ are equal to $\pi/2$, then $F(\mu/\alpha)$ is infinite and equation (19) (giving the temperature distribution in the fin) is invalid. For $n = 2$ and $L/W = 22.98$, β_2 is equal to β_1 and a_3 given by equation (31) is not defined. Consequently a_4–a_9 expressed in equations (32)–(37) are not predictable and equation (19) is again invalid. Thus in the case where $n = 1$, $N = 0.25$, $Bi = 0.01$ and $L/W > 27.58$ and in the case where $n = 2$, $N = 0.25$, $Bi = 0.01$ and $L/W > 22.98$, the fin fails to function, i.e. $T_b < 1$. The foregoing implies that N_m, the maximum value of the generation number, is equal to 0.25 for $n = 1$, $Bi = 0.01$ and $L/W = 27.58$ and for $n = 2$, $Bi = 0.01$ and $L/W = 22.98$.

If the hypothetical boundary condition at the fin tip is considered, the asymptotic value of T_b is equal to the inverse ratio of the square root of N for $n = 1$ and to the inverse ratio of the cubic root of N for $n = 2$ for all values of Bi [25]. This boundary condition implies that no heat transfer takes place at the fin tip. Accordingly the fin is infinitely long and therefore for given values of n, Bi, L/W and N ($0 < N \leqslant 1$), the fin is always able to transfer all the heat generated in it and all the heat conducted into it at its base to its surroundings. But if heat transfer takes place at the fin tip then the fin length should be finite since the boundary condition expressed in equation (4) should be fulfilled. For given values of Bi and N (i.e. $0 < N \leqslant N_m$), this results in that either the fin does

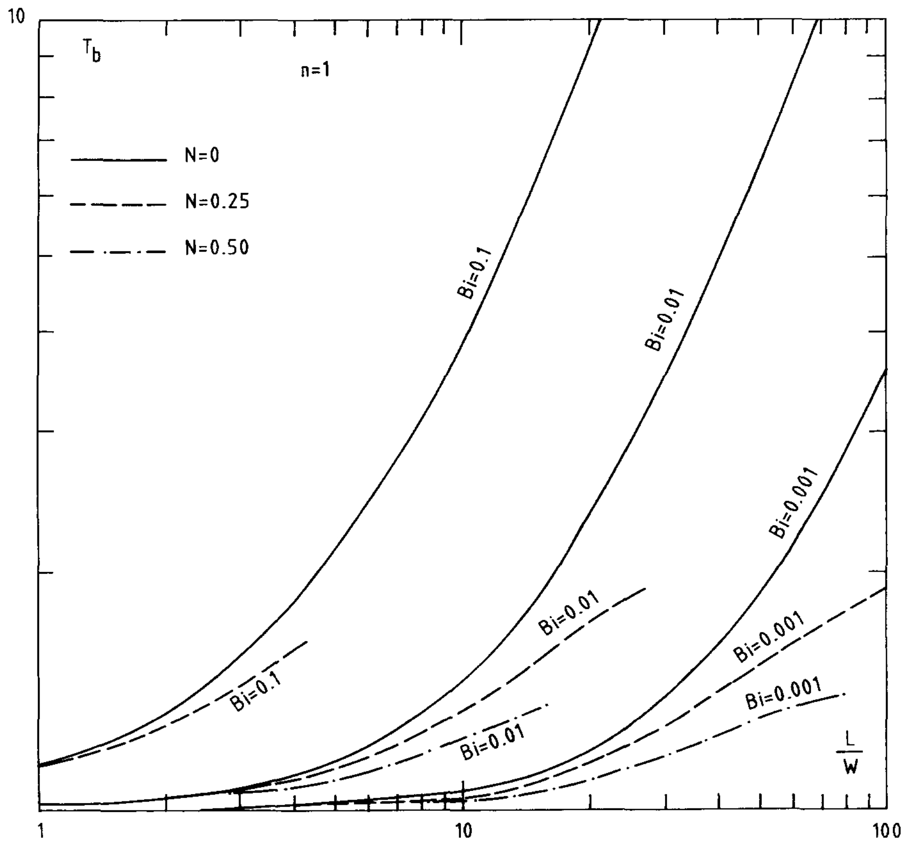

FIG. 2. Dimensionless temperature at the fin base for $n = 1$.

not function at all or that it operates only if the value of L/W is smaller than a certain value. Beyond this certain value, the fin is invalid, i.e. $T_b < 1$. These are in fact foregone conclusions since Bi shows the ratio of the interior to exterior resistances and the heat transfer coefficient decreases along the fin if $n = 1$ and 2. This is further illustrated in Figs. 2 and 3: the smaller the value of N the larger the maximum value of L/W; the smaller the value of Bi the larger the maximum value of L/W; and the smaller the value of n the larger the maximum value of L/W.

That which has been stated above also implies that the assumption that the heat transfer coefficient is constant yields incorrect solutions for $0 < N \leqslant N_m$, if in fact this coefficient is inconstant and heat transfer takes place at the fin tip. It follows from Fig. 1 or equation (46) that T_b has an asymptotic value for a given value of Bi if $n = 0$, contrary to the results obtained for $n = -1$, 1 and 2.

The foregoing clearly shows that the solutions of equation (2) for the fin with uniform internal heat generation are trivial for some values of parameters considered if either the hypothetical boundary condition at the fin tip is assumed or a uniform heat transfer coefficient is assumed whilst this coefficient is nonuniform. There is, however, another undesirable feature in the solution of equation (2) with this boundary condition. This is quantitatively described in the following section.

THE EFFECT ON THE PERFORMANCE OF A FIN OF FIN TIP BOUNDARY CONDITION

In order to evaluate this effect, the rate of heat transfer from a straight (or a cylindrical) fin to the fluid surrounding it is taken as a criterion. This rate of heat transfer is expressed as

$$q = -AK\left(\frac{d\theta}{dx}\right)_{x=0} + QAL. \qquad (47)$$

The first term on the right-hand side of equation (47) gives the rate of heat flow into the fin at its base and the second term the rate of heat flow due to internal heat generation. Dividing equation (47) by $A\theta_b h_b$ yields the non-dimensional rate of heat transfer from the fin to its surroundings

$$S = f + N\frac{L}{W} \qquad (48)$$

where

$$S = \frac{q}{Ah_b\theta_b} \qquad (49)$$

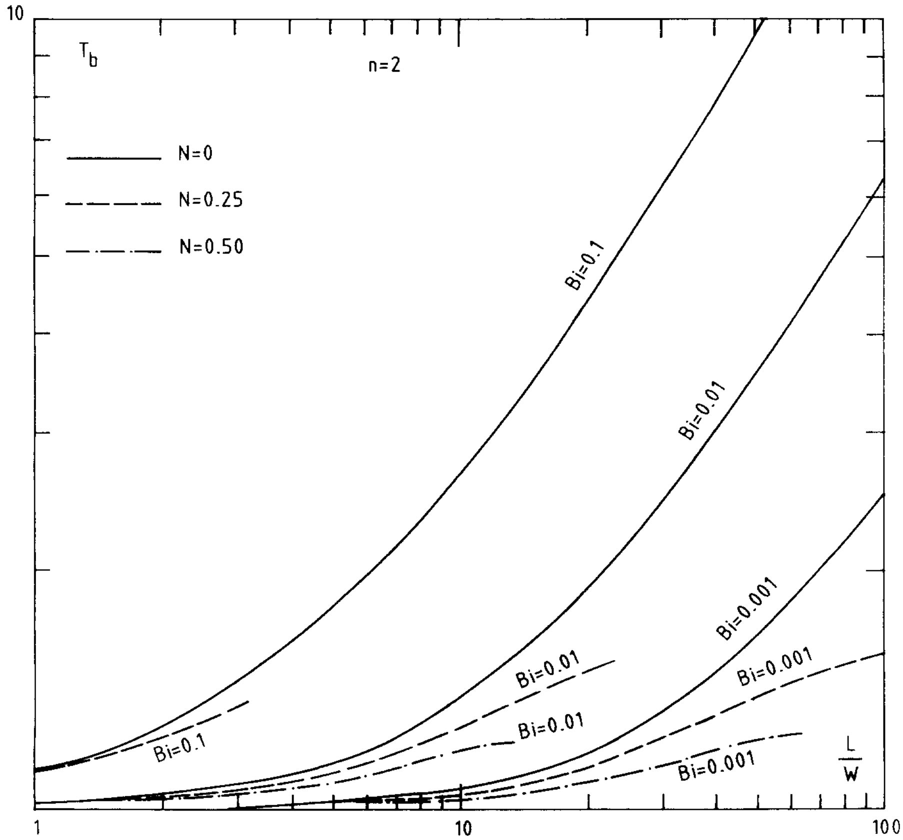

FIG. 3. Dimensionless temperature at the fin base for $n = 2$.

$$f = \frac{-K\left(\frac{\mathrm{d}\theta}{\mathrm{d}x}\right)_{x=0}}{h_b\theta_b}. \qquad (50)$$

The value of f given by equation (50) is, in fact, the fin effectiveness which gives the ratio of heat fluxes with and without fin on a surface. Replacing $(\mathrm{d}\theta/\mathrm{d}x)$ in equation (50) with the dimensionless temperature gradient; using the expression given by equation (17) for the latter; considering $T = T_b$ for $X = 0$; and after rearranging, equation (50) yields f for all values of n, excluding $n = -2$

$$f = \frac{1}{\sqrt{Bi}}\left(\frac{2}{(n+2)T_b^{(n+2)}}\left\{T_b^{(n+2)} - (n+2)NT_b^{(n+2)} + (n+2)NT_b^{(n+1)} - 1 + \frac{(n+2)Bi}{2T_b^n}\right\}\right)^{0.5}. \qquad (51)$$

If no internal heat sources exist in the fin, all the terms including N in equation (51) disappear (i.e. $N = Q = 0$). If the hypothetical boundary condition at the fin tip is considered, the last term including Bi in equation (51) also disappears [25]. f is a function of n, Bi, N and T_b and T_b that of n, Bi, N and L/W.

For given values of n, Bi, L/W and N, the dimensionless rates of heat transfer from the straight (or the cylindrical) fin were calculated with equation (48) considering the real and the hypothetical boundary conditions. The error in the calculation of the rate of heat transfer from the fin to its surroundings using the hypothetical boundary condition at the fin tip is defined by

$$e = \left|\frac{S_h - S_t}{S_t}\right| \cdot 100. \qquad (52)$$

Taking Bi and N as exemplary parameters, this error was plotted against L/W for $n = -1$ and 0 in Fig. 4, and for $n = 1$ and 2 in Figs. 5 and 6, respectively. The value of L/W was varied between 1 and 100 and N was taken equal to 0 and 0.5 and Bi to 0.001, 0.01 and 0.1. These ranges of the foregoing quoted parameters are of practical importance [1, 2, 22].

For $n = -1$, the heat flux on the surface of the fin is constant. Therefore, equation (53), the derivation of which is a straightforward matter, can also be used to determine e, thus

$$e = \left|-100\frac{1}{L/W+1}\right|. \qquad (53)$$

It is obvious from equation (53) that the error is not a function of Bi and N if $n = -1$.

The following conclusions have been drawn from

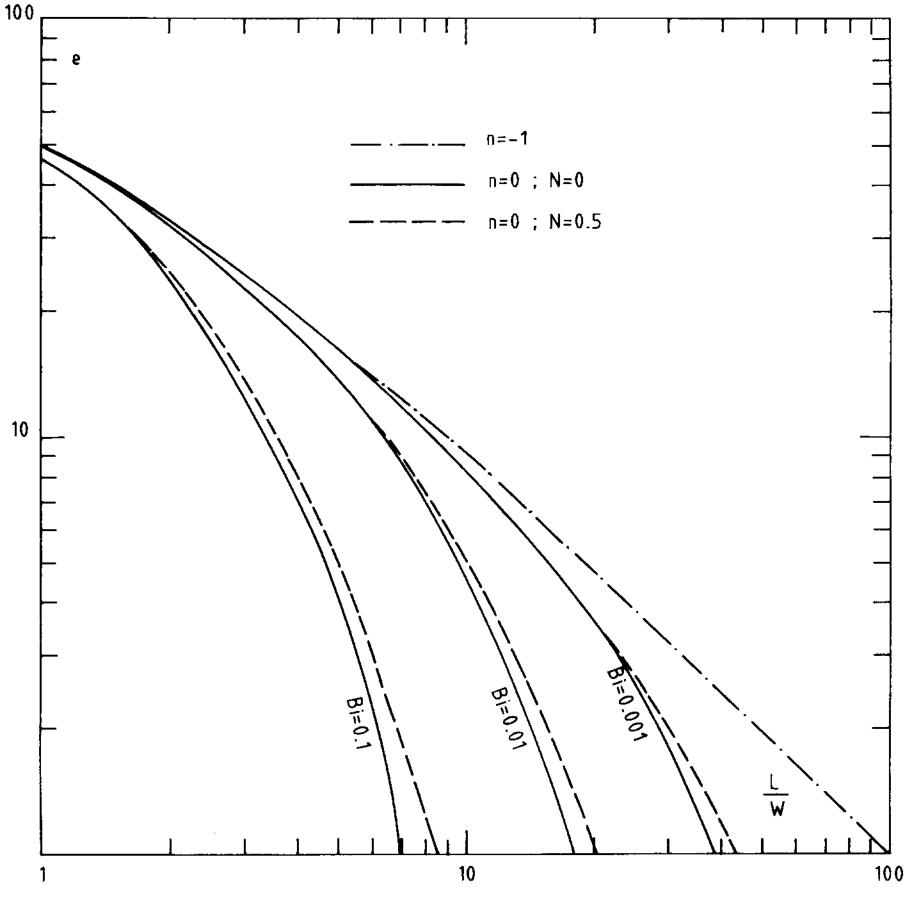

FIG. 4. Percentage error in the determination of the rate of heat flow from a fin to its surroundings for $n = -1$ and 0.

Figs. 4–6. For given values of n (excluding $n = -1$), Bi and L/W, the effect on the error of internal heat generation seems to be quite insignificant and it becomes negligibly small if n increases. For given values of Bi, L/W and N, the effect on e of n (excluding $n = -1$) seems to be of second-order importance at least for high values of Bi.

For $n = 0$, 1, 2 and small values of Bi (i.e. $Bi < 0.001$), e approaches that given for $n = -1$ in Fig. 4. This is explained as follows: the physical meaning of Bi implies that for a sufficiently small value of Bi and a given value of L/W, the temperature distribution in the fin should be very flat (e.g. see equations (15) and (16)). This means a practically constant heat flux on the surface of the fin, a condition which applies to the case where $n = -1$. When $n = 0$ for example and beyond $Bi \leqslant 10^{-7}$, e is identical to the value given in Fig. 4 for $n = -1$.

It follows from Figs. 5 and 6 that for $Bi = 0.01$ and $N = 0.5$, the fin fails to function if $L/W > 14.8382$ for $n = 1$ and if $L/W > 11.1460$ for $n = 2$. For $Bi = 0.1$, $N = 0.5$ and $n = 1$ or 2 the fin is again invalid for $L/W \geqslant 1$.

The curve given in Fig. 4 for when $n = 0$, and when $N = 0$ applied to the most studied case in the literature. In accordance to this curve, the error varies between 50 and 10% for $Bi = 0.1$, 0.01 and 0.001 if $0.88 \leqslant L/W \leqslant 3.51$, $0.98 \leqslant L/W \leqslant 6.51$ and $1.00 \leqslant L/W \leqslant 8.51$, respectively. The foregoing ranges of parameters seem to be of importance for practical applications. For $L/W > 9$, $e < 10\%$. Methods were proposed to improve the results obtained with the solution of equation (2) for a straight fin [3] and a cylindrical fin [5] if the hypothetical boundary condition at the fin tip was considered. In accordance with the practice of these methods, L in this solution should be replaced by $(L+W)$ in order to take into account the heat losses from the fin tip. Accordingly the equations presented herein should first be reduced to the condition in which no heat transfer takes place at the fin tip, as explained previously. Thereafter it is sufficient to replace L in equations (7) and (48) by $(L+W)$. In this case the error is less than 2% approximately for all the range of parameters being considered in Figs. 4–6 if $N = 0$. If $N = 0.5$ and $1 \leqslant L/W \leqslant 100$, and where $n = 0$, $Bi = 0.001$, 0.01 and 0.1 and where $Bi = 0.001$, 0.01 and $n = 1$ or 2, the error is less than 8.3% approximately.

For $N > 0$, given values of Bi and L/W and $n = 1$

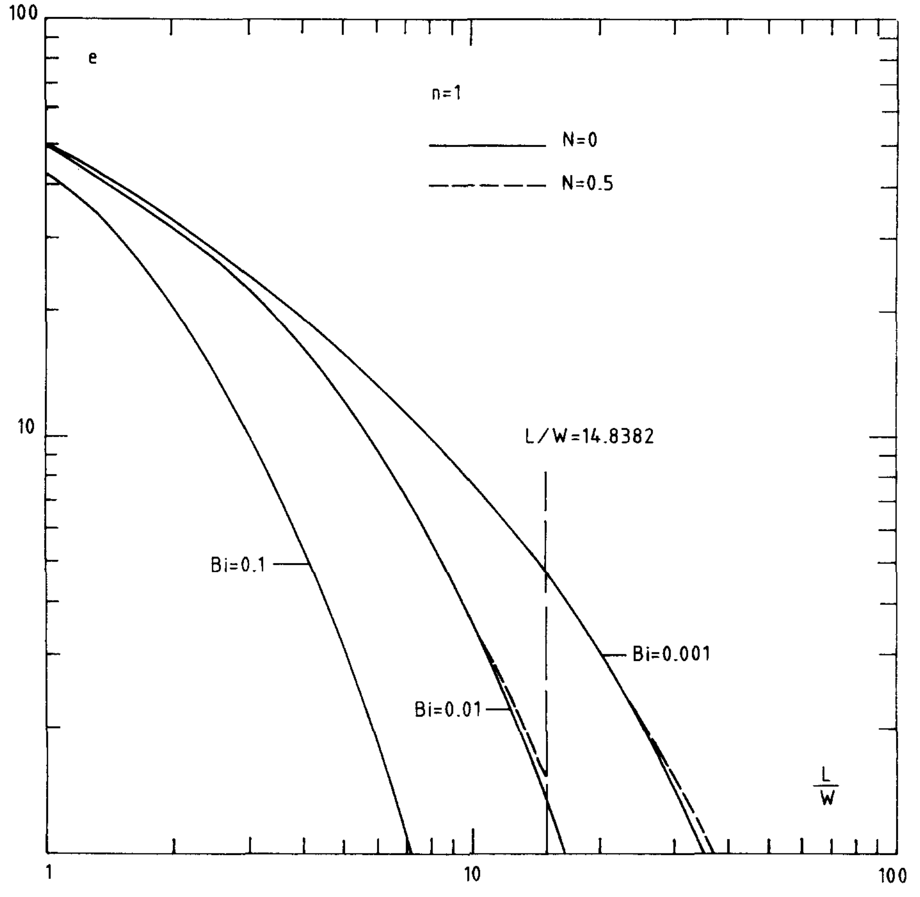

FIG. 5. Percentage error in the determination of the rate of heat flow from a fin to its surroundings for $n = 1$.

or 2, the corrected length should be used with caution since the roots of the polynomial equation in equation (17) should fulfill the conditions required for them if T_b determined with the corrected length is used in this polynomial equation. Otherwise this value of T_b is trivial (or the fin is invalid).

In the literature there is a widely used fin criterion, i.e. the fin efficiency. In order to obtain the fin efficiency the fin effectiveness should be divided by $(L/W+1)$ and (L/W) if the real and the hypothetical boundary conditions at the fin tip are considered, respectively.

SUMMARY/CONCLUSIONS

The analytic solutions for the one-dimensional temperature distribution in a straight (or a cylindrical) fin with and without internal heat generation are presented on the assumption that there are two different boundary conditions at the fin tip, i.e. the condition where heat transfer takes place at the fin tip and the condition where no heat transfer takes place there. The heat transfer coefficient is a power function of the temperature difference between the fin and its surroundings. The exponent in this power function is taken as being equal to -1, 0, 1 and 2. The ranges of parameters considered for the quantitative analysis of the fin are: the modified Biot number Bi, 0.1 – any value smaller than 0.1; the generation number N, 0–0.5; the ratio of the fin length to the half-fin thickness (or the half-radius) L/W, 1–100. These ranges of conditions are appropriate for the one-dimensional analysis of the fin.

If it is assumed that no heat transfer takes place at the fin tip, then the results obtained for the foregoing ranges of parameters indicate that the determination of the rate of heat transfer from the fin to its surroundings includes a fairly large error for some conditions which are important for practical applications, and that the solutions obtained for the temperature distributions in the fin are trivial beyond some values of the parameters considered if internal heat sources exist in the fin.

The foregoing error in the determination of the rate of heat transfer from the fin is less than 10% for $L/W > 9$, and for $L/W \leqslant 9$ this error seems to be acceptable if a so-called corrected fin length is adopted.

If heat transfer takes place at the fin tip and the heat transfer coefficient is assumed to be uniform whilst in

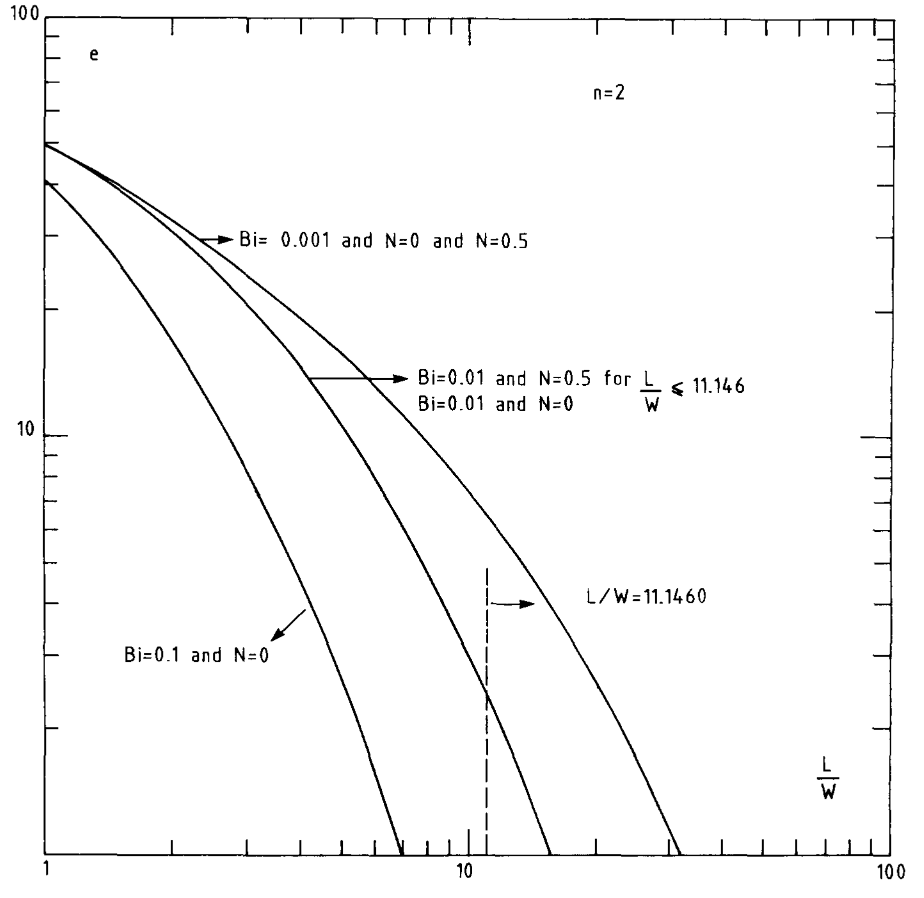

FIG. 6. Percentage error in the determination of the rate of heat flow from a fin to its surroundings for $n = 2$.

fact it is nonuniform, then the temperature distribution obtained in a fin with uniform heat generation is trivial beyond the same values of parameters considered.

For the calculation of Legendre's normal elliptic integral of the first kind, simple formulae are presented and which are needed if the power in the quoted power function is equal to 1 and 2.

To the knowledge of the author and for these particular powers, the analytic solutions of the foregoing quoted differential equation for a straight (or a cylindrical) fin with uniform internal heat generation are not available in the literature if heat transfer takes place at the fin tip. This differential equation is also of practical significance in chemical and nuclear engineering.

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APPENDIX: SIMPLE FORMULAE FOR THE DETERMINATION OF $F(\mu/\alpha)$

These formulae are presented herein for the convenience of the design engineer, as can be deduced from the following: $F(\mu/\alpha)$, Legendre's normal elliptic integral of the first kind, is an infinite series [28] and does not rapidly converge for large values of α. The value of μ may vary between 0 and π and α between 0 and $\pi/2$. For the adequate determination of $F(\mu/\alpha)$ (i.e. to obtain the tabulated values of $F(\mu/\alpha)$), it is sufficient to consider the first 4, 12 and 121 terms in this infinite series for $\alpha = \pi/12$, $3\pi/12$ and $5\pi/12$, respectively. The value of $F(\mu/\alpha)$ is tabulated for $0 < \mu \leqslant \pi/2$ whilst values of $F(\mu/\alpha)$ for amplitudes greater than $\pi/2$ are required for the determination of the temperature distribution in a straight (or cylindrical) fin where $n = 1$ and 2. If $89\pi/180 < \alpha < \pi/2$ and $89\pi/180 < \mu < \pi/2$, then the tabulated values of $F(\mu/\alpha)$ can only be used if the value of α is reduced with a suitable transformation.

For $0 \leqslant \mu \leqslant \pi$ and $0 \leqslant \alpha \leqslant \pi/12$, $F(\mu/\alpha)$ is given by

$$(F/\alpha) = \mu + J_1^2(0.5J_2 + 0.375J_1^2J_3 + 0.3125J_1^4J_4) \tag{A1}$$

where

$$J_1 = \arcsin\alpha \tag{A2}$$

$$J_2 = 0.50(\mu - \sin\mu\cos\mu) \tag{A3}$$

$$J_3 = 0.75(J_2 - \tfrac{1}{3}\sin^3\mu\cos\mu) \tag{A4}$$

$$J_4 = \tfrac{5}{6}(J_3 - 0.2\sin^5\mu\cos\mu). \tag{A5}$$

Equation (A1) predicts $F(\mu/\alpha)$ with an error of less than approximately 0.003%. The error is based on a nine-digit value of $F(\mu/\alpha)$.

If $0 \leqslant \mu \leqslant \pi/2$ and $\pi/12 < \alpha < \pi/2$, the values of α and μ should be reduced using the Gauss transformation [28]. Let

$$J_5 = \sqrt{(1-\sin^2\alpha)} \tag{A6}$$

$$J_6 = (1-J_5)/(1+J_5). \tag{A7}$$

The reduced amplitude and modular angle are then given by

$$\mu_r = \arcsin\left(\frac{1-\sqrt{(1-\sin^2\alpha\sin^2\mu)}}{(1-J_5)\sin\mu}\right) \tag{A8}$$

$$\alpha_r = \arcsin J_6 \tag{A9}$$

and the following relation holds good between $F(\mu/\alpha)$ and $F(\mu_r/\alpha_r)$:

$$F(\mu/\alpha) = (1+J_6)F(\mu_r/\alpha_r). \tag{A10}$$

If the reduced modular angle is still higher than $\pi/12$, then the values of α_r and μ_r are successively reduced using equations (A6)–(A10) until the last reduced modular angle is equal to $\pi/12$ or smaller than $\pi/12$, for which equation (A1) is valid. For $\mu = \alpha = \pi/2$, $F(\mu/\alpha)$ is infinite, and for $\mu = \pi/2$ and $0 < \alpha < \pi/2$, the value of μ cannot be reduced but only the value of α as equations (A6)–(A9) imply.

If $\pi/2 < \mu \leqslant \pi$ and $\pi/12 < \alpha < \pi/2$, $F(\mu/\alpha)$ is divided into two parts [14]

$$F(\mu/\alpha) = 2F(\{\pi/2\}/\alpha) - F(\{\pi-\mu\}/\alpha). \tag{A11}$$

Since $(\pi-\mu)$ is smaller than $\pi/2$, the previously described method is used to determine $F(\{\pi-\mu\}/\alpha)$ and $F(\{\pi/2\}/\alpha)$, and $F(\mu/\alpha)$ is predicted with equation (A11).

EFFET DE LA CONDITION LIMITE AU SOMMET D'UNE AILETTE SUR LA PERFORMANCE DE L'AILETTE AVEC OU SANS GENERATION INTERNE DE CHALEUR

Résumé—Une étude analytique est conduite pour une ailette droite monodimensionnelle (ou cylindrique) avec ou sans génération de chaleur, en considérant l'effet de l'hypothèse qu'il n'y a pas de transfert de chaleur à l'extrémité. L'erreur apparait plus grande pour certaines conditions importantes pour les applications pratiques. Voici les domaines de variation des paramètres: nombre de Biot: 0,1 et valeurs inférieures; nombre de génération: 0–0,5 et rapport de la longueur à la demi-épaisseur de l'ailette (ou au demi-rayon): 1–100. Le coefficient de transfert de chaleur est supposé être une fonction puissance de la différence de température entre l'ailette et l'environnement et la puissance est égale à −1, 0, 1 et 2.

DER EINFLUSS DER RANDBEDINGUNG AN EINER RIPPEN-SPITZE AUF DIE LEISTUNGSFÄHIGKEIT DER RIPPE MIT UND OHNE INNERE WÄRMEERZEUGUNG

Zusammenfassung—Die Wärmeabgabe einer eindimensionalen geraden (oder zylindrischen) Rippe mit und ohne innere Wärmeerzeugung wurde unter der Annahme, daß an der Rippen-Spitze kein Wärmeübergang stattfindet, analytisch untersucht. Diese Annahme liefert für einige Werte der betrachteten Parameter triviale Lösungen, wenn innere Wärmequellen in der Rippe existieren. Der Fehler bei der Bestimmung der obigen Wärmeabgabe erweist sich für einige praktisch wichtige Bedingungen als bedeutend. Für die quantitative Analyse der Rippe werden die folgenden Parameter-Bereiche benutzt: Biot-Zahl: kleiner oder gleich 0,1; Wärmeerzeugungs-Zahl: 0–0,5; Verhältnis von Rippenlänge zu ihrer halben Dicke (oder ihrem halben Radius): 1–100. Der Wärmeübergangskoeffizient wird als eine Potenzfunktion der Temperaturdifferenz zwischen der Rippe und ihrer Umgebung angenommen; die Exponenten in dieser Funktion sind −1; 0; 1 und 2.

ВЛИЯНИЕ ГРАНИЧНОГО УСЛОВИЯ НА КОНЦЕ РЕБРА НА ЕГО РАБОЧИЕ ХАРАКТЕРИСТИКИ ПРИ НАЛИЧИИ И В ОТСУТСТВИЕ ВНУТРЕННЕГО ТЕПЛОВЫДЕЛЕНИЯ

Аннотация—Выполнены аналитические исследования влияния одномерного прямого (или цилиндрического) ребра с внутренним тепловыделением и без него на скорость теплового потока в предположении, что на конце ребра не происходит теплообмена. Данное предположение приводит к тривиальным решениям для некоторых рассматриваемых значений параметров при наличии в ребре внутренних тепловых источников. Ошибка в определении скорости теплового потока оказывается большой для некоторых условий, существенных для практического применения. В количественном анализе ребра рассматривались следующие диапазоны параметров: число Био: 0,1—любое значение ниже 0,1; число тепловыделения: 0–0,5 и отношение длины ребра к его полутолщине (или к полурадиусу): 1–100. Коэффициент теплопереноса определяется из зависимости энергии от разности температур ребра и окружающей среды, причем энергия равна −1, 0, 1 и 2.